The Zakharov-Shabat spectral problem on the semi-line: Hilbert formulation and applications

2001 J. Phys. A: Math. Gen. 34 7359

(http://iopscience.iop.org/0305-4470/34/36/316)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 7359–7380 PII: S0305-4470(01)24414-X

The Zakharov–Shabat spectral problem on the semi-line: Hilbert formulation and applications

J Leon and A Spire

Physique Mathématique et Théorique, CNRS-UMR5825, Université Montpellier 2, 34095 Montpellier, France

Received 1 May 2001
Published 31 August 2001
Online at stacks.iop.org/JPhysA/34/7359

Abstract

The inverse spectral transform for the Zakharov–Shabat equation on the semi-line $x > 0$ is reconsidered as a Hilbert problem. The boundary data induce an essential singularity as $k \to \infty$ to one of the basic solutions. Then solving the inverse problem means solving a Hilbert problem with particular prescribed behaviour. It is demonstrated that the direct and inverse problems are solved in a consistent way as soon as the spectral transform vanishes as $\mathcal{O}(1/k)$ at infinity in the whole upper half-plane (where it may possess single poles) and is continuous and bounded on the real k axis. The method is applied to stimulated Raman scattering and sine-Gordon (light cone) for which it is demonstrated that time evolution conserves the properties of the spectral transform.

PACS numbers: 02.30.Jr, 41.20.-q

1. Introduction

Since the discovery of the inverse spectral transform (IST) in 1967 [1] to solve nonlinear evolutions (Korteveg–de Vries and later nonlinear Schrödinger [2], then sine-Gordon (SG), etc [3]) with Cauchy conditions on the infinite space (and vanishing asymptotic boundary values), the application of the method to boundary value problems on semi-infinite (or finite) intervals has been the subject of intense research with few general results, which is reminiscent of the fact that, even for linear partial differential equations, boundary value problems greatly differ from one another. It would be useless to quote all the papers that have dealt with boundary value problems for nonlinear evolutions, so only the most recent ones will be cited here (see [4, 25] where more complete references will be found).

Taking for instance the nonlinear Schrödinger equation (NLS) [2] for a field $q(x, t)$, saying that it is solvable by IST on the infinite line actually means that, given Dirichlet condition $q(x, 0)$ on the line $x \in (-\infty, +\infty)$ and Cauchy conditions $q(\pm\infty, t) = 0$ and $q_x(\pm\infty, t) = 0$ for any $t > 0$, the IST method allows us to construct the solution $q(x, t)$ at any t for all x. Although Dirichlet conditions at $x = \pm\infty$, namely $q(\pm\infty, t) = 0$, constitute with $q(x, 0)$

a well-posed problem, the solution of NLS with IST *requires* the adjunction of the Neuman condition $q_x(\pm\infty, t) = 0$. This simply results from the fact that the Lax pair contains explicitly the derivative q_x. The point is that this supplementary constraint is compatible with the rest of the boundary values and remains compatible at any time. This *stability* is a property of the spectral transform itself that conserves the class of function (potentials) chosen at $t = 0$. For instance, if $q(x, 0)$ is in the Schwartz space (the function and all its derivatives are continuous and vanish faster than any power of x as $x \to \infty$), then the solution $q(x, t)$ obtained by IST is also in the Schwartz space which makes the method work.

Such is no longer the case on a finite (or semi-infinite) interval. Indeed, while the constraint $q_x(\pm\infty, t) = 0$ is not restrictive when we already have $q(\pm\infty, t) = 0$, it is not so on a finite domain where imposing the data of both the function and its derivative produce an over-constrained problem. Then, although NLS keeps many of its *integrability properties* on the finite interval (with mixed Dirichlet–Neumann conditions on the semi-axis) [4, 5], and Bäcklund transformation can be used to determine explicit solvable subclasses [6], the general problem is still unsolved.

A detailed study of the Dirichlet boundary problem (data of $q(x, 0)$, $q(0, t)$ and $q(\infty, t) = 0$) has been performed in [10] where it has been shown that the problem of the determination of $q_x(0, t)$ from $q(0, t)$ is transformed into the problem of determining some *missing spectral datum*. This approach allows one also to construct classes of solutions but not to solve the problem in general.

Another point of view is presented in [11] where the method is based on the scattering problem on a line in the (x, t)-plane in the Gelfand–Levitan–Marchenko formulation. This is applied to the Korteveg–de Vries equation and also provides much insight into the boundary-value problem but shows that indeed the problem is still open.

An interesting approach has been proposed in [13] where both operators of the Lax pair are treated as joint spectral problems. There the boundary-value problem becomes a Dirichlet condition on a closed domain, namely data of $q(x, 0)$, $q(0, t)$, $q(\infty, t) = 0$ and $q(\mathrm{X}, T)$. Applied to NLS (defocusing case), this method provides the solution as a system of coupled integral equations which is then difficult to use. But this approach allows for particular solutions, see also [12], and for large time asymptotic behaviours. Moreover it has been successfully applied to SG in the light cone [13] for which the Lax pair is compatible with the boundary values, and then extended to boundary values on an arbitrary convex domain [14]. The method has also proved very useful to treat linear partial differential equations with moving boundaries [15]. The method furnishes the solution under the form of a set of Cauchy–Green integral equations.

There is another case where the general solution has been obtained by IST: the stimulated Raman scattering equations (SRS) on the semi-axis [16] (the *sharp-line* version on the finite interval has been treated in [17] by the method of [13]). The boundary-value problem there is an open-end problem and the stability of direct/inverse spectral transformation allows us to get the general solution on the semi-axis (as well as the finite interval case) which has been successfully checked in numerical simulations [18]. The method also works for the discrete integrable version of SRS [19] and it is based on the approach of the *dispersion relation* [20] where the principal Lax operator is taken as the spectral operator and the auxiliary Lax operator is the tool to define *altogether* the dispersion relation and the time evolution of the spectral transform. The advantage of such a method is that, in the physically relevant case of a medium initially at rest, it allows one to get all the information on the nonlinear Fourier spectrum explicitly from the input radiation fields. An interesting consequence is the possibility to build literally the spectral transform out of measurements and compare it to the theory, which has been done successfully in numerical simulations [18] of the finite interval, as well as experimental data [21] with infinite line damped model. The method has also been extended to treat perturbation of SRS [22], confirming the validity of approximations used for comparison with experiments in [21].

As a general result, it has been shown that the dispersion relation will depend explicitly not only on the boundary values (as for an infinite line [23]) but also on the spectral transform itself, hence producing nonlinear time evolution of spectral data (of Riccati type) [16]. Interestingly enough, such an evolution induces the motion of the poles (discrete eigenvalues of the spectral problem, associated with solitons) and thus results in creation (and annihilation) of solitons due to input boundaries. The same property has been found for the Karpman–Kaup system [24] with a remarkable illustration by numerical simulations [25].

The spectral problem underlying SRS and SG is the Zakharov–Shabat system on the semi-axis with prescribed potential values in $x = 0$. We show here that the solution can be written out of a system integral equation of Cauchy type where the contour of integration differs from the one working in the infinite line case. This results from the presence of an essential singularity in the asymptotic behaviours of one of the Jost solutions, a problem that has been underestimated in the literature so far.

Then we shall begin with solving the direct and inverse spectral problems in sections 2 and 3. In doing this we shall discover that the reconstructed solutions (Jost solutions) must obey a sequence of properties, named P1–P9, that are central to the method. This is then applied to SRS in section 4 and to SG in section 5. In both cases it is demonstrated that the time evolution of the spectral transform is such that it conserves the constraints necessary to ensure the nine fundamental properties of the Jost solutions. In order to illustrate the method for the SG, we consider the explicitly solvable case of a piecewise constant boundary datum which generically stands for the discretization of an arbitrary boundary value.

2. The direct problem

2.1. Basic solutions

The time t being an external parameter, it is omitted in this section. We consider the Zakharov–Shabat spectral problem [2] on the semi-line

$$x > 0 : \phi_x = \mathrm{i}k[\phi, \sigma_3] + Q(x)\phi \qquad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \tag{2.1}$$

with given boundary values

$$q(x)|_{x=0^+} = q_0 \qquad r(x)|_{x=0^+} = r_0. \tag{2.2}$$

Jost solutions [26] can be defined as the solution of the integral equations

$$\begin{pmatrix} \psi^+_{11}(k,x) \\ \psi^+_{21}(k,x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^x \mathrm{d}y\, q(y)\psi^+_{21}(k,y) \\ \int_0^x \mathrm{d}y\, r(y)\psi^+_{11}(k,y)\mathrm{e}^{2\mathrm{i}k(x-y)} \end{pmatrix} \tag{2.3}$$

$$\begin{pmatrix} \phi^+_{12}(k,x) \\ \phi^+_{22}(k,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\int_x^\infty \mathrm{d}y\, q(y)\phi^+_{22}(k,y)\mathrm{e}^{-2\mathrm{i}k(x-y)} \\ \int_0^x \mathrm{d}y\, r(y)\phi^+_{12}(k,y) \end{pmatrix} \tag{2.4}$$

$$\begin{pmatrix} \phi^-_{11}(k,x) \\ \phi^-_{21}(k,x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^x \mathrm{d}y\, q(y)\phi^-_{21}(k,y) \\ -\int_x^\infty \mathrm{d}y\, r(y)\phi^-_{11}(k,y)\mathrm{e}^{2\mathrm{i}k(x-y)} \end{pmatrix} \tag{2.5}$$

$$\begin{pmatrix} \psi^-_{12}(k,x) \\ \psi^-_{22}(k,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_0^x \mathrm{d}y\, q(y)\psi^-_{22}(k,y)\mathrm{e}^{-2\mathrm{i}k(x-y)} \\ \int_0^x \mathrm{d}y\, r(y)\psi^-_{12}(k,y) \end{pmatrix}. \tag{2.6}$$

2.2. Boundary behaviours

The so-called *physical solution*:

$$\Phi(k,x) = (\phi_1^-, \phi_2^+) \tag{2.7}$$

obeys the bounds

$$\Phi(k,x)|_{x=0^+} = \begin{pmatrix} 1 & \rho^+ \\ \rho^- & 1 \end{pmatrix} \tag{2.8}$$

$$\Phi(k,x)|_{x\to\infty} = \begin{pmatrix} \tau^- & 0 \\ 0 & \tau^+ \end{pmatrix} \tag{2.9}$$

where the *reflection coefficients* $\rho^\pm(k)$ are defined by reading the above integral equations at $x = 0^+$, namely

$$\begin{aligned} \rho^+(k) &= -\int_0^\infty \mathrm{d}y\, q(y)\phi_{22}^+(k,y)\mathrm{e}^{2\mathrm{i}ky} \\ \rho^-(k) &= -\int_0^\infty \mathrm{d}y\, r(y)\phi_{11}^-(k,y)\mathrm{e}^{-2\mathrm{i}ky} \end{aligned} \tag{2.10}$$

and the *transmission coefficients* $\tau^\pm(k)$ by making the limit as $x \to \infty$, i.e.

$$\begin{aligned} \tau^+(k) &= 1 + \int_0^\infty \mathrm{d}y\, r(y)\phi_{12}^+(k,y) \\ \tau^-(k) &= 1 + \int_0^\infty \mathrm{d}y\, q(y)\phi_{21}^-(k,y). \end{aligned} \tag{2.11}$$

By direct computation with (2.1) we demonstrate that the determinant of any solution is independent of x and hence can be calculated at $x = 0$ or at $x \to \infty$. Applied to Φ it gives

$$1 - \rho^+\rho^- = \tau^+\tau^- \tag{2.12}$$

which is called the *unitarity relation*.

The rest of the solution, namely

$$\Psi(k,x) = (\psi_1^+, \psi_2^-) \tag{2.13}$$

is used to solve the inverse problem and we call it the *intermediate solution*. It obeys the following bounds for $k \in \mathbb{R}$:

$$\Psi(k,x)|_{x=0^+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.14}$$

$$\Psi(k,x)|_{x\to\infty} \sim \begin{pmatrix} 1/\tau^+ & -\mathrm{e}^{-2\mathrm{i}kx}\rho^+/\tau^+ \\ -\mathrm{e}^{2\mathrm{i}kx}\rho^-/\tau^- & 1/\tau^- \end{pmatrix} \tag{2.15}$$

(the boundary behaviour as $x \to \infty$ is actually obtained by using the relations that follow). The intermediate solution is related to the physical solution by the following relations for $k \in \mathbb{R}$:

$$\begin{aligned} \psi_1^+ - \phi_1^- &= -\mathrm{e}^{2\mathrm{i}kx}\rho^-\psi_2^- \\ \phi_2^+ - \psi_2^- &= \mathrm{e}^{-2\mathrm{i}kx}\rho^+\psi_1^+ \end{aligned} \tag{2.16}$$

which can be proved in different ways, for instance by the computation of the differences out of (2.4) and (2.6) and the comparison of the resulting integral equation with the one for the rhs of (2.16). A simpler but equivalent method is to express the vector solution ϕ_1^- on the basis ψ_1^+ and $e^{2ikx}\psi_2^-$. The coefficients of this expansion are then calculated by comparing the boundary values in $x = 0^+$. Another proof is by expressing the solution Ψ in terms of Φ as

$$\Phi(k,x) = \Psi(k,x)\, e^{-ik\sigma_3 x} \begin{pmatrix} 1 & \rho^+(k) \\ \rho^-(k) & 1 \end{pmatrix} e^{ik\sigma_3 x} \tag{2.17}$$

(computed by matching the values in $x = 0^+$) which in particular readily provides the behaviour (2.15) from (2.9).

2.3. Analytical properties

Using standard method [26] one obtains that, for potentials vanishing fast enough as $x \to \infty$, the vector ψ_1^+ is analytic in the upper half k-plane and ψ_2^- in the lower half. Next, the vector ϕ_2^+ is a meromorphic function of k in the upper half-plane with N^+ simple poles k_n^+ which are the zeros of the analytic function $1/\tau^+(k)$. Indeed from (2.15) we have

$$\frac{1}{\tau^+} = 1 + \int_0^\infty q\psi_{21}^+ \tag{2.18}$$

with ψ_{21}^+ analytic. Moreover the vector

$$\tilde{\phi}_2^+ = \frac{1}{\tau^+}\phi_2^+ \tag{2.19}$$

can be written from (2.4) as the solution of

$$\begin{pmatrix} \tilde{\phi}_{12}^+(k,x) \\ \tilde{\phi}_{22}^+(k,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_0^x dy\, q(y)\tilde{\phi}_{22}^+(k,y)e^{-2ik(x-y)} \\ \int_0^x dy\, r(y)\tilde{\phi}_{12}^+(k,y) \end{pmatrix} \tag{2.20}$$

which shows that $\tilde{\phi}_2^+$ is analytic in k and thus that ϕ_2^+ has poles where the analytic function $1/\tau^+(k)$ has zeros.

Similarly, ϕ_1^- is meromorphic in the lower half-plane with N^- poles k_n^-, zeros of $1/\tau^-(k)$. Consequently, from their very definitions (2.10) and (2.11), $\tau^\pm$ and $\rho^\pm$ have meromorphic extensions in their respective half-planes $\pm\text{Im}\,(k) > 0$ where they possess the $N^\pm$ simple poles $k_n^\pm$ (the bound states locations). In particular we have the relation, obtained simply by taking the residues on (2.17),

$$\underset{k_n^+}{\text{Res}}\{\phi_2^+\} = \rho_n^+\psi_1^+(k_n^+)\exp[-2ik_n^+x] \qquad \rho_n^+ = \underset{k_n^+}{\text{Res}}\{\rho^+\} \tag{2.21}$$

$$\underset{k_n^-}{\text{Res}}\{\phi_1^-\} = \rho_n^-\psi_2^-(k_n^-)\exp[2ik_n^-x] \qquad \rho_n^- = \underset{k_n^-}{\text{Res}}\{\rho^-\}. \tag{2.22}$$

The spectral data is then constituted by the set

$$\mathcal{S} = \{\text{Im}\,(k) > 0,\ \rho^+(k);\ \text{Im}\,(k) < 0,\ \rho^-(k)\} \tag{2.23}$$

which is complete if we prove that it allows us to reconstruct the potentials $r(x)$ and $q(x)$ with $q(0^+) = q_0$ and $r(0^+) = r_0$.

2.4. Large k behaviours

In order to solve the Hilbert problem (2.16) with the singularities $k_n^\pm$ in the complex plane, one must calculate first the behaviour of the solutions $\Phi(k,x)$ and $\Psi(k,x)$ as $|k| \to \infty$. Note that we follow Mushkhelishvili [27] in referring to (2.16) as a Hilbert problem.

By using repeatedly integration by parts we obtain from the integral equations (2.4) and (2.6) the following regular asymptotic behaviour of the physical solution:

$$\begin{aligned}(\phi_1^-,\phi_2^+) = \mathbf{1} &+ \frac{1}{2ik}\begin{pmatrix} -\int_0^x rq & q \\ -r & \int_0^x rq \end{pmatrix} + \frac{1}{(2ik)^2} \\ &\times \begin{pmatrix} -\int_0^x [r_y q - rq\int_0^y rq] & -q_x + q\int_0^x rq \\ -r_x + r\int_0^x rq & -\int_0^x [rq_y - rq\int_0^y rq] \end{pmatrix} + \mathcal{O}\left(\frac{1}{k^3}\right)\end{aligned} \tag{2.24}$$

which has to be understood in Im $(k) \leqslant 0$ for the first vector ϕ_1^- and in Im $(k) \geqslant 0$ for ϕ_2^+.

The same procedure applied to the intermediate solution shows that it possess an essential singularity at $k = \infty$ on the real axis as indeed we get

$$
\begin{aligned}
(\psi_1^+, \psi_2^-) = \mathbf{1} + \frac{1}{2ik}\begin{pmatrix} -\int_0^x rq & q \\ -r & \int_0^x rq \end{pmatrix} + \frac{1}{2ik}\begin{pmatrix} 0 & -q_0 e^{-2ikx} \\ r_0 e^{2ikx} & 0 \end{pmatrix} \\
+ \frac{1}{(2ik)^2}\begin{pmatrix} -\int_0^x [r_y q - rq\int_0^y rq] & -q_x + q\int_0^x rq \\ -r_x + r\int_0^x rq & -\int_0^x [rq_y - rq\int_0^y rq] \end{pmatrix} \\
+ \frac{1}{(2ik)^2}\begin{pmatrix} (qe^{2ikx} - q_0)r_0 & (q_0\int_0^x rq + q_0')e^{-2ikx} \\ (r_0\int_0^x rq + r_0')e^{2ikx} & (re^{-2ikx} - r_0)q_0 \end{pmatrix} + \cdots .
\end{aligned} \tag{2.25}
$$

Consequently, when q_0 and r_0 do not vanish, the intermediate solution does not have the same behaviour as the physical solution, in contrast with the infinite line case. In particular the formulation of the solution of the Hilbert problem will differ from the one of the infinite line.

Note for future use that from the definitions (2.10) of $\rho^\pm$ follow the large k expansions:

$$
\begin{aligned}
\rho^+(k) &= \frac{1}{2ik}\, q_0 - \frac{1}{(2ik)^2} q_0' + \frac{1}{(2ik)^3}(q_0'' - q_0^2 r_0) + \mathcal{O}\left(\frac{1}{k^4}\right) \\
\rho^-(k) &= -\frac{1}{2ik}\, r_0 - \frac{1}{(2ik)^2} r_0' - \frac{1}{(2ik)^3}(r_0'' - r_0^2 q_0) + \mathcal{O}\left(\frac{1}{k^4}\right)
\end{aligned} \tag{2.26}
$$

where we have defined

$$
q_0' = \left.\frac{\partial q(x)}{\partial x}\right|_{x=0^+} \qquad q_0'' = \left.\frac{\partial^2 q(x)}{\partial x^2}\right|_{x=0^+} \tag{2.27}
$$

so as for $r(x)$.

3. Solution of the Hilbert problem

3.1. Statement of the problem

Let us be given the set of *spectral data*:

$$
\mathcal{S} = \{\text{Im}\,(k) \geqslant 0,\ \rho^+(k);\ \text{Im}\,(k) \leqslant 0,\ \rho^-(k)\} \tag{3.1}
$$

where $\rho^+(k)$ (resp. $\rho^-(k)$) is continuous and bounded on the real axis and meromorphic in the upper (resp. lower) half plane with a finite number of single poles k_n^+ (resp. k_n^-) and related residues ρ_n^+ (resp. ρ_n^-). Moreover they obey in their respective half-planes

$$
\rho^+(k) = \frac{1}{2ik}\, q_0 + \mathcal{O}\left(\frac{1}{k^2}\right) \qquad \rho^-(k) = -\frac{1}{2ik}\, r_0 + \mathcal{O}\left(\frac{1}{k^2}\right) \tag{3.2}
$$

where q_0 and r_0 are given numbers. The problem is to construct the two matrix $\Phi(k, x)$ and $\Psi(k, x)$ solution of the Hilbert problem (2.16) obeying the following set of properties.

Property P.1. Alternative expression of the Hilbert problem:

$$
\Phi = \Psi M \qquad M = e^{-ik\sigma_3 x}\begin{pmatrix} 1 & \rho^+ \\ \rho^- & 1 \end{pmatrix} e^{ik\sigma_3 x} \qquad x \geqslant 0.
$$

Property P.2. The vector ψ_1^+ (resp. ψ_2^-) is analytic in Im $(k) > 0$ (resp. Im $(k) < 0$). The vector ϕ_2^+ (resp. ϕ_1^-) is meromorphic in Im $(k) > 0$ (resp. Im $(k) < 0$) where it possesses N^+ (resp. N^-) simple poles k_n^+ (resp. k_n^-) with residues

$$
\underset{k_n^+}{\text{Res}}\{\phi_2^+\} = \rho_n^+ \psi_1^+(k_n^+) \exp[-2ik_n^+ x] \qquad \underset{k_n^-}{\text{Res}}\{\phi_1^-\} = \rho_n^- \psi_2^-(k_n^-) \exp[2ik_n^- x].
$$

Property P.3. The functions $\Psi(k,x)$ and $\Phi(k,x)$ obey the bounds

$$\Psi(k,0)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \Phi(k,0)=\begin{pmatrix}1 & \rho^+(k)\\ \rho^-(k) & 1\end{pmatrix}.$$

Property P.4. The function $\Psi(k,x)-\mathbf{1}$ vanishes for $x<0$.

Property P.5. The function $\Phi(k,x)$ has the large k expansion

$$\Phi(k,x)=\mathbf{1}+\frac{1}{k}\Phi^{(1)}(x)+\mathcal{O}(1/k^2)$$

(in the lower half-plane for the first vector, in the upper half-plane for the second).

Property P.6. The function $\Psi(k,x)$ has an expansion with an essential singularity at $k=\infty$ on the real axis

$$\Psi=\mathbf{1}+\frac{1}{k}\Psi^{(1)}(x)+\frac{1}{k}\Psi_0^{(1)}\mathrm{e}^{2\mathrm{i}k\sigma_3x}+\cdots.$$

Property P.7. There exists an off-diagonal matrix $Q(x)$ such that for $x>0$

$$\Psi_x=\mathrm{i}k[\Psi,\sigma_3]+Q(x)\Psi$$

which is called the Zakharov–Shabat spectral equation.

Property P.8. The function Q obeys

$$Q(x)|_{x=0^+}=\begin{pmatrix}0 & q_0\\ r_0 & 0\end{pmatrix} \qquad Q(x)|_{x=0^-}=0 \qquad Q(x)\underset{x\to\infty}{\longrightarrow}0.$$

Property P.9. The above defined function $Q(x)$ is also given by

$$Q(x)=\mathrm{i}[\sigma_3,\Phi^{(1)}(x)].$$

3.2. The solution

We prove now that the solution of the Hilbert problem (2.16) obeying properties P.1–P.9 above is given by the following equations. First, $\Phi(k,x)$ is given by the explicit expressions

$$\phi_1^-(k,x)=\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{2\mathrm{i}\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0}\frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^-(\lambda)\mathrm{e}^{2\mathrm{i}\lambda x}\psi_2^-(\lambda,x)$$
$$-\sum_{n=1}^{N^-}\frac{\rho_n^-}{k_n^--k}\psi_2^-(k_n^-,x)\mathrm{e}^{2\mathrm{i}k_n^-x} \qquad \mathrm{Im}\,(k)\leqslant 0 \tag{3.3}$$

$$\phi_2^+(k,x)=\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2\mathrm{i}\pi}\int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0}\frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^+(\lambda)\mathrm{e}^{-2\mathrm{i}\lambda x}\psi_1^+(\lambda,x)$$
$$-\sum_{n=1}^{N^+}\frac{\rho_n^+}{k_n^+-k}\psi_1^+(k_n^+,x)\mathrm{e}^{-2\mathrm{i}k_n^+x} \qquad \mathrm{Im}\,(k)\geqslant 0 \tag{3.4}$$

where the function $\Psi(k,x)$ solves the following Cauchy–Green integral system of equations:

$$\psi_1^+(k,x)=\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{2\mathrm{i}\pi}\int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0}\frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^-(\lambda)\mathrm{e}^{2\mathrm{i}\lambda x}\psi_2^-(\lambda,x)$$
$$-\sum_{n=1}^{N^-}\frac{\rho_n^-}{k_n^--k}\psi_2^-(k_n^-,x)\mathrm{e}^{2\mathrm{i}k_n^-x} \qquad \mathrm{Im}\,(k)\geqslant 0 \tag{3.5}$$

$$\psi_2^-(k,x)=\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2\mathrm{i}\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0}\frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^+(\lambda)\mathrm{e}^{-2\mathrm{i}\lambda x}\psi_1^+(\lambda,x)$$
$$-\sum_{n=1}^{N^+}\frac{\rho_n^+}{k_n^+-k}\psi_1^+(k_n^+,x)\mathrm{e}^{-2\mathrm{i}k_n^+x} \qquad \mathrm{Im}\,(k)\leqslant 0. \tag{3.6}$$

In contrast with what occurs in the one-dimensional case $x \in (-\infty, +\infty)$, here the integrals do not run on the real axis but on contours along the real axis in the upper or lower half-plane as indicated. We shall see that this apparently slight difference has a lot of consequences and allows in particular reconstruction of the very solutions Φ and Ψ defined in the direct problem.

It is worth stressing here that we do not address the problem of the *complete characterization* of the spectral data $\rho^{\pm}(k)$, namely to find the *necessary and sufficient* conditions which would ensure a unique solution of the above Cauchy–Green integral equations (such a problem has been worked out in [12]). However, we shall work with a set of sufficient conditions on $\rho^{\pm}(k)$, ensuring that the reconstructed potential does coincide with the one we started with. In particular for what concerns solution of nonlinear evolutions, we will demonstrate that a set of actual spectral data (constructed from a given potential at time 0) evolve in time such as to conserve this set of sufficient conditions.

We proceed now step by step to demonstrate that the expressions (3.3)–(3.6) do produce functions Φ and Ψ that verify properties P.1–P.9.

Property P.1. For real values of k the integral equations (3.5) and (3.6) can be written by virtue of the Cauchy theorem (by assumption $\rho^{\pm}(k)$ have no poles on the real axis):

$$\psi_1^+(k,x) = \begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2\mathrm{i}\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^-(\lambda)\mathrm{e}^{2\mathrm{i}\lambda x}\psi_2^-(\lambda,x) \\ -\rho^-(k)\mathrm{e}^{2\mathrm{i}kx}\psi_2^-(k,x) - \sum_{n=1}^{N^-}\frac{\rho_n^-}{k_n^- - k}\psi_2^-(k_n^-,x)\mathrm{e}^{2\mathrm{i}k_n^- x} \qquad k\in\mathbb{R} \tag{3.7}$$

$$\psi_2^-(k,x) = \begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2\mathrm{i}\pi}\int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^+(\lambda)\mathrm{e}^{-2\mathrm{i}\lambda x}\psi_1^+(\lambda,x) \\ -\rho^+(k)\mathrm{e}^{-2\mathrm{i}kx}\psi_1^+(k,x) - \sum_{n=1}^{N^+}\frac{\rho_n^+}{k_n^+ - k}\psi_1^+(k_n^+,x)\mathrm{e}^{-2\mathrm{i}k_n^+ x} \qquad k\in\mathbb{R}. \tag{3.8}$$

Hence thus follow the Hilbert relations

$$\psi_1^+ - \phi_1^- = -\mathrm{e}^{2\mathrm{i}kx}\rho^-\psi_2^- \qquad \phi_2^+ - \psi_2^- = \mathrm{e}^{-2\mathrm{i}kx}\rho^+\psi_1^+ \tag{3.9}$$

and by rearrangement of the vectors into matrices, property P.1 is proved.

Property P.2. The analytical properties of the various vectors follow from their very expressions. Indeed, as the set of single poles of Φ is explicitly displayed, one has only to prove that the Cauchy-like integrals produce holomorphic functions [27]. Such is the case if the integrant is a continuous bounded function of $\lambda \in \mathbb{R}$ vanishing as $\lambda \to \infty$. This actually results from the assumed spectral data $\rho^{\pm}(\lambda)$ and from the large $k \in \mathbb{R}$ properties of $\Psi(k, x)$ deduced from the integral equations (3.5) and (3.6) as property P.6 which follows.

Property P.3. The equations (3.5) and (3.6) evaluated at $x = 0$ (x is a parameter) lead to the system

$$\psi_1^+(k,0) = \begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2\mathrm{i}\pi}\int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^-(\lambda)\psi_2^-(\lambda,0) - \sum_{n=1}^{N^-}\frac{\rho_n^-}{k_n^- - k}\psi_2^-(k_n^-,0) \tag{3.10}$$

$$\psi_2^-(k,0) = \begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2\mathrm{i}\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^+(\lambda)\psi_1^+(\lambda,0) - \sum_{n=1}^{N^+}\frac{\rho_n^+}{k_n^+ - k}\psi_1^+(k_n^+,0). \tag{3.11}$$

Now, as we assume a *unique solution* to the integral equations, it is sufficient to verify that $\Psi(k, 0) = \mathbf{1}$ is indeed a solution of the above system. For $\psi_1^+(k, 0)$ we then have the equation

$$\psi_1^+(k,0) = \begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2\mathrm{i}\pi}\int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda-k}\,\rho^-(\lambda)\begin{pmatrix}0\\1\end{pmatrix} - \sum_{n=1}^{N^-}\frac{\rho_n^-}{k_n^- - k}\begin{pmatrix}0\\1\end{pmatrix}$$

which by contour integration in the lower half plane, and thanks to the properties of ρ^-, easily gives $\psi_1^+(k,0) = (1,0)^T$ (remember that k lives in the upper half-plane). The same procedure applies to the second vector and gives $\psi_2^-(k,0) = (0,1)^T$.

Next, by using exactly the same method with $\Phi(k,0)$, and taking care of the pole $\lambda = k$, we obtain the property P.3 (note that one could also use property P.1 to derive the boundary value of one of the functions from the other one).

Property P.4. For $x < 0$, thanks to the factor $\mathrm{e}^{2\mathrm{i}\lambda x}$ appearing in the expression (3.5), we can perform the integration in the lower half-plane for ψ_1^+. The contribution of the poles exactly cancels out and $\psi_1^+(k,x) = (1,0)^T$ for $x < 0$. The same procedure holds for ψ_2^- which proves property P.4.

Property P.5. The expansion of $\Phi(k,x)$ at large k readily results from the expressions (3.3) and (3.4) where the Cauchy integral behaves well. Indeed, taking e.g. ϕ_1^-, we integrate above the real axis where, for $x > 0$, the exponential is bounded.

We obtain in particular for the matrix $\Phi^{(1)}(x) = (\phi_1^{(1)}, \phi_2^{(1)})$

$$\phi_1^{(1)} = \frac{1}{2\mathrm{i}\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \rho^-(\lambda)\mathrm{e}^{2\mathrm{i}\lambda x}\psi_2^-(\lambda,x) + \sum_{n=1}^{N^-}\rho_n^-\psi_2^-(k_n^-,x)\mathrm{e}^{2\mathrm{i}k_n^- x} \tag{3.12}$$

$$\phi_2^{(1)} = -\frac{1}{2\mathrm{i}\pi}\int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0} \rho^+(\lambda)\mathrm{e}^{-2\mathrm{i}\lambda x}\psi_1^+(\lambda,x) + \sum_{n=1}^{N^+}\rho_n^+\psi_1^+(k_n^+,x)\mathrm{e}^{-2\mathrm{i}k_n^+ x}. \tag{3.13}$$

Property P.6. The expansion of $\Psi(k,x)$ at large $k \in \mathbb{R}$ cannot be obtained directly from the integral equations (3.5) and (3.6) (one sees on (3.5) that $\exp[2\mathrm{i}\lambda x]$ is integrated for $x > 0$ on a contour lying in the lower half λ-plane), but rather form the versions (3.7) and (3.8) where the Cauchy integrals behave well. Then the same procedure as for property P.5 applies to obtain the expansion given in property P.6 (it is actually easier to use properties P.5 and P.1 together with the expansion of the spectral data $\rho^\pm(k)$ to demonstrate property P.6). We obtain in particular, according to the notations in property P.6,

$$\Psi^{(1)}(x) = \Phi^{(1)}(x) \qquad \Psi_0^{(1)} = \begin{pmatrix} 0 & -\frac{q_0}{2\mathrm{i}} \\ \frac{r_0}{2\mathrm{i}} & 0 \end{pmatrix} \tag{3.14}$$

which matches the expressions (2.25) (and can be pursued to next orders).

Property P.7. This is a fundamental property in the process of solving the inverse problem as indeed it gives the expression of the potential from the solution of the Cauchy–Green integral equations (3.5) and (3.6): in other words, in terms of the spectral data $\rho^\pm(k)$.

To perform the proof of property P.7 it is first more convenient to rewrite the integral equations as follows:

$$\psi_1^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2\mathrm{i}\pi}\int_{C_-}\frac{\mathrm{d}\lambda}{\lambda - k}\rho^-\mathrm{e}^{2\mathrm{i}\lambda x}\psi_2^- \tag{3.15}$$

$$\psi_2^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\mathrm{i}\pi}\int_{C_+}\frac{\mathrm{d}\lambda}{\lambda - k}\rho^+\mathrm{e}^{-2\mathrm{i}\lambda x}\psi_1^+ \tag{3.16}$$

where C_+ is a smooth contour in the upper half-plane extending from $-\infty + \mathrm{i}0$ to $\infty + \mathrm{i}0$ and passing *over* all poles k_n^+ (respectively for C_- passing *under* all poles k_n^-). Note that ψ_2^- is written for $\mathrm{Im}(k) \leqslant 0$ and hence the pole $\lambda = k \in \mathbb{R}$ does not contribute to the integral in the expression (3.16) because it runs entirely in the lower half-plane. The same holds oppositely for ψ_1^+. In short these integral equations are

$$\Psi(k,x) = \mathbf{1} + \frac{1}{2\mathrm{i}\pi}\int_C \frac{\mathrm{d}\lambda}{\lambda - k}\Psi(\lambda,x)R(\lambda,x) \tag{3.17}$$

where

$$R(k,x) = \mathrm{e}^{-\mathrm{i}k\sigma_3 x}\begin{pmatrix} 0 & \rho^+(k) \\ -\rho^-(k) & 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}k\sigma_3 x} \tag{3.18}$$

and where the contour of integration has to be understood as being C_- for the first vector of the product ΨR, and C_+ for the second.

Thanks to the particular dependence in x of $R(k,x)$ it is easy to compute $\Psi_x + \mathrm{i}k[\sigma_3, \Psi]$ and by algebraic manipulations to get

$$\Psi_x + \mathrm{i}k[\sigma_3,\Psi] = -\frac{1}{2\pi}\int_C \mathrm{d}\lambda\,[\sigma_3, \Psi R] + \frac{1}{2\mathrm{i}\pi}\int_C \frac{\mathrm{d}\lambda}{\lambda - k}\left(\Psi_x + \mathrm{i}\lambda[\sigma_3,\Psi]\right)R. \tag{3.19}$$

Then by defining the off-diagonal matrix

$$Q(x) = -\frac{1}{2\pi}\int_C \mathrm{d}\lambda\,[\sigma_3, \Psi(\lambda,x)R(\lambda,x)] \tag{3.20}$$

we can compare the integral equation (3.19) to the integral equation for the matrix $Q(x)\Psi(\lambda,x)$. As the solution is assumed to be unique we arrive at the desired result

$$\Psi_x + \mathrm{i}k[\sigma_3,\Psi] = Q\Psi. \tag{3.21}$$

Property P.8. We derive now from the above definition of $Q(x)$ the property P.8 and to that end rewrite (3.20) as

$$\begin{aligned} q(x) &= -\frac{1}{\pi}\int_{C_+}\mathrm{d}\lambda\,\rho^+(\lambda)\psi_{11}^+(\lambda,x)\mathrm{e}^{-2\mathrm{i}\lambda x} \\ r(x) &= -\frac{1}{\pi}\int_{C_-}\mathrm{d}\lambda\,\rho^-(\lambda)\psi_{22}^-(\lambda,x)\mathrm{e}^{2\mathrm{i}\lambda x}. \end{aligned} \tag{3.22}$$

First, for $x < 0$, we can close the contours with the circle of infinite radius in the related half-planes and readily obtain that q and r vanish (remember that the contours pass beyond the poles of ρ).

To evaluate the boundary values in $x = 0$ we use $\psi_{11}^+(k,0) = \psi_{22}^-(k,0) = 1$ (the function Ψ is continuous in $x = 0$), but keep the exponentials as they produce discontinuous functions in $x = 0$. Indeed, with the help of the large k behaviours (3.2) we may write

$$\lim_{x\to 0^+} q(x) = \lim_{x\to 0^+}\left\{-\frac{q_0}{2\mathrm{i}\pi}\int_{C_+}\mathrm{d}\lambda\,\frac{\mathrm{e}^{-2\mathrm{i}\lambda x}}{\lambda}\right\} - \frac{1}{\pi}\int_{C_+}\mathrm{d}\lambda\left[\rho^+(\lambda) - \frac{q_0}{2\mathrm{i}\lambda}\right]. \tag{3.23}$$

With the formulae

$$x > 0 \;\Rightarrow\; -\frac{1}{2\mathrm{i}\pi}\int_{C_+}\mathrm{d}\lambda\,\frac{\mathrm{e}^{-2\mathrm{i}\lambda x}}{\lambda} = \frac{1}{2\mathrm{i}\pi}\int_{C_-}\mathrm{d}\lambda\,\frac{\mathrm{e}^{2\mathrm{i}\lambda x}}{\lambda} = 1 \tag{3.24}$$

we obtain $q(0^+) = q_0$ and $r(0^+) = r_0$, where q_0 and r_0 are given from the spectral data $\rho^\pm$ in (3.2). Since the function $\rho^+(\lambda) - q_0/2\mathrm{i}\lambda$ is analytic above C_+ and vanishes as $1/k^2$, we also obtain $q(0^+) = q_0$, where q_0 is given from the spectral data ρ^+ in (3.2). The same is true for $r(x)$ and we get $r(0^+) = r_0$.

Note that the procedure can be applied to compute the x derivative of the potential by first regularizing it in $x = 0$ and by using the next order expansion of $\rho(k)$ given in (2.26). Indeed we have

$$\frac{\partial}{\partial x}\{q(x) - q_0\theta(x)\} = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_{C_+} \mathrm{d}\lambda \left[\rho^+\psi_{11}^+ - \frac{q_0}{2\mathrm{i}\lambda}\right]\mathrm{e}^{-2\mathrm{i}\lambda x} \tag{3.25}$$

and applying the same procedure we obtain eventually

$$\lim_{x\to 0^+}\left\{\frac{\partial}{\partial x}q(x)\right\} = \frac{\partial}{\partial x}\{q(x) - q_0\theta(x)\}\big|_{x=0} = q_0'. \tag{3.26}$$

The large x vanishing behaviours of q and r follow from their expressions (3.22) rewritten as (we consider only $q(x)$: the same computation works as well for $r(x)$)

$$q(x) = -\frac{1}{\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \mathrm{d}\lambda\, \rho^+(\lambda)\psi_{11}^+(\lambda, x)\mathrm{e}^{-2\mathrm{i}\lambda x} + 2\mathrm{i}\sum_{n=1}^{N^+}\rho_n^+\psi_{11}^+(k_n^+, x)\mathrm{e}^{-2\mathrm{i}k_n^+ x}. \tag{3.27}$$

The above expression vanishes at large x because the path integral vanishes and because

$$\psi_{11}^+(k, x) \underset{x\to\infty}{\longrightarrow} \frac{1}{\tau^+(k)} \qquad \left.\frac{1}{\tau^+(k)}\right|_{k=k_n^+} = 0 \tag{3.28}$$

which results from (2.15) and from the very definition of the poles k_n^+.

Property P.9. Results from the fact that, due to properties P.1 and P.7, the function Φ also solves Zakharov–Shabat, i.e.

$$\Phi_x = \mathrm{i}k[\Phi, \sigma_3] + Q(x)\Phi. \tag{3.29}$$

Then by inserting in it the large k behaviour given by property P.5 we must arrive at property P.9. To check consistency we have then to demonstrate that the following two expressions for $q(x)$:

$$-\frac{1}{\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \mathrm{d}\lambda\, \rho^+(\lambda)\psi_{11}^+(\lambda, x)\mathrm{e}^{-2\mathrm{i}\lambda x} + 2\mathrm{i}\sum_{n=1}^{N^+}\rho_n^+\psi_{11}^+(k_n^+, x)\mathrm{e}^{-2\mathrm{i}k_n^+ x} \tag{3.30}$$

$$-\frac{1}{\pi}\int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0} \mathrm{d}\lambda\, \rho^+(\lambda)\psi_{11}^+(\lambda, x)\mathrm{e}^{-2\mathrm{i}\lambda x} + 2\mathrm{i}\sum_{n=1}^{N^+}\rho_n^+\psi_{11}^+(k_n^+, x)\mathrm{e}^{-2\mathrm{i}k_n^+ x} \tag{3.31}$$

do coincide. This simply results from the starting hypothesis of a spectral transform $\rho(k)$ which is continuous and bounded on the real axis (no poles). Then both the path integrals above are equal.

Note that the relation property P.9 giving $Q(x)$ can be evaluated in $x = 0$, which does lead to the expected result thanks to the boundary behaviour (2.8) of Φ in $x = 0$ and the large k behaviours of ρ^+ and ρ^-. Note also that, in order to compute $q(0^+)$, one should first deform the contour to pass *above* all poles k_n^+ and then make use of (3.24).

3.3. Reductions

The two particular evolutions that we shall study then result from a *reduction* from the two potentials r and q to only one. The first reduction concerns SRS and is

$$\overline{Q} = \sigma_2\, Q\, \sigma_2 \;\Leftrightarrow r = -\bar{q}. \tag{3.32}$$

Then by direct computation one easily proves that $\sigma_2\overline{\Psi(\bar{k}, x)}\sigma_2$ solves the same spectral equation (2.1) as $\Psi(k, x)$ and possess the same boundary behaviour (2.14) in $x = 0$. Those two functions then coincide:

$$\Psi(k, x) = \sigma_2\overline{\Psi(\bar{k}, x)}\sigma_2 \tag{3.33}$$

and examination of their boundary behaviour as $x \to \infty$ gives the reduction constraint on the spectral transform:

$$\overline{\tau^-(\overline{k})} = \tau^+(k) \qquad \overline{\rho^-(\overline{k})} = -\rho^+(k). \tag{3.34}$$

Considering the poles and residues, the above expressions imply

$$N^+ = N^- \qquad \overline{\rho_n^-} = -\rho_n^+ \qquad \overline{k_n^-} = k_n^+. \tag{3.35}$$

The second case is related to SG in the light cone. It is

$$Q = \sigma_2 Q \sigma_2 \;\Leftrightarrow r = -q. \tag{3.36}$$

In that case it is easy to prove that

$$\Psi(k, x) = \sigma_2 \Psi(-k, x)\sigma_2 \tag{3.37}$$

and examination of the boundary behaviour as $x \to \infty$ and $x \to 0$ gives the reduction constraint on the spectral transform

$$\begin{aligned} &\rho^+(k, t) = -\rho^-(-k, t) \qquad \tau^-(-k) = \tau^+(k) \\ &N^+ = N^- \qquad \rho_n^- = -\rho_n^+. \end{aligned} \tag{3.38}$$

In the case of SG, we also need to implement the reality constraint $q \in \mathbb{R}$ for which both relations (3.32) and (3.36) have to hold together. As a consequence, by direct manipulation of relations (3.34) and (3.38), the resulting constraint on the spectral transform $\rho = \rho^+$ which comes in addition to (3.38) is

$$\overline{\rho}(-\overline{k}) = \rho(k). \tag{3.39}$$

As a consequence the poles in the complex plane come in pairs, namely if k_n is a pole of $\rho(k)$ with residue ρ_n, then $-\overline{k}_n$ is also a pole with residue $-\overline{\rho}_n$. In short

$$\operatorname*{Res}_{k_n}\{\rho(k)\} = \rho_n \;\Rightarrow\; \operatorname*{Res}_{-\overline{k}_n}\{\rho(k)\} = -\overline{\rho}_n. \tag{3.40}$$

4. The Dirichlet problem for SRS

4.1. Boundary values and Lax pair

The formalism of the direct and inverse problem for the Zakharov–Shabat system on the semi-line $x > 0$ is now applied to the SRS equations for the pulse envelopes of the pump $a(k, x, t)$, the Stokes $b(k, x, t)$ and the medium excitation $q(x, t)$:

$$\begin{aligned} &\partial_x a = q b \mathrm{e}^{2\mathrm{i}kx} \qquad \partial_x b = -\overline{q} a \mathrm{e}^{-2\mathrm{i}kx} \\ &\partial_t q = -\int g(k)\, \mathrm{d}k\; a\overline{b}\mathrm{e}^{-2\mathrm{i}kx} \end{aligned} \tag{4.1}$$

where $g(k)$ measures the coupling (related to Raman gain) and the parameter k is a mismatch wavenumber resulting from group velocity dispersion [16]. The Lax pair for SRS has been written in the case $g(k) = \delta(k - k_0)$ in [28] and regularized by means of the parameter k in [20] which found its physical meaning in [16].

Dirichlet conditions are prescribed on the line $x = 0$, $t > 0$:

$$a(k, x, t)|_{x=0} = A(t) \qquad b(k, x, t)|_{x=0} = B(t) \tag{4.2}$$

and on the line $x > 0$, $t = 0$:

$$q(x, t)|_{t=0} = c(x). \tag{4.3}$$

Note that we could assume boundary values $A(k,t)$ and $B(k,t)$ instead of $A(t)$ and $B(t)$, but here we have chosen to leave the k dependence in the arbitrary distribution $g(k)$. This is equivalent to assuming separable dependence of A an B in k and t. The given boundary value $c(x)$ is assumed to be of Schwartz type (it decreases at large x with all its derivatives faster than any polynomial) but the physically relevant case is a medium initially at rest, namely $c(x) = 0$. Note that the compatibility of this set of Dirichlet boundary data is inferred from the system (4.1) which gives

$$\partial_x a(k,x,0)|_{x=0} = c(0)B(0) \qquad \partial_x b(k,x,0)|_{x=0} = -\overline{c}(0)A(0). \tag{4.4}$$

The above Dirichlet problem has been solved in [16] and, for sake of completeness, we give hereafter the main lines of the derivation of the solution. The vector $(a, b\mathrm{e}^{2\mathrm{i}kx})^T$ is expanded on the basis ψ_1^+ and $\phi_2^+\mathrm{e}^{2\mathrm{i}kx}$, the coefficients being obtained by examination of the values in $x = 0$. The result is

$$\begin{pmatrix} a \\ b\mathrm{e}^{2\mathrm{i}kx} \end{pmatrix} = (A - \rho B)\psi_1^+ + B\phi_2^+. \tag{4.5}$$

As a consequence we have

$$\begin{pmatrix} a \\ b \end{pmatrix} \underset{x\to\infty}{\longrightarrow} \begin{pmatrix} (A - \rho B)/\tau \\ \overline{\rho}(A - \rho B)/\overline{\tau} + B\tau \end{pmatrix}. \tag{4.6}$$

The time dependence is plugged into the spectral data by requiring that the solution $\mu^+ = (\psi_1^+, \phi_2^+)$ of the Zakharov–Shabat equation also be a solution for $k \in \mathbb{R}$ of

$$\begin{aligned} &\mu_t^+ = V^+\mu^+ + \mu^+\mathrm{e}^{-\mathrm{i}k\sigma_3 x}\Omega^+\mathrm{e}^{\mathrm{i}k\sigma_3 x} \\ &V^+ = \frac{1}{4\mathrm{i}}\int \frac{g(\lambda)\,\mathrm{d}\lambda}{\lambda - (k + \mathrm{i}0)} \begin{pmatrix} |a|^2 - |b|^2 & 2a\overline{b}\mathrm{e}^{-2\mathrm{i}\lambda x} \\ 2\overline{a}b\mathrm{e}^{2\mathrm{i}\lambda x} & |b|^2 - |a|^2 \end{pmatrix}. \end{aligned} \tag{4.7}$$

Note that, as the matrix $V(k,x,t)$ is discontinuous when k crosses the real axis, we have to choose a given half-plane. A similar equation could be written for $\mu^- = (\phi_1^-, \psi_2^-)$ with V^- and Ω^- but, due to the reduction symmetry, it is redundant. The compatibility between the two operators (3.29) and (4.7) with the constraint $\Omega_x = 0$ results in the equation (4.1) within the reduction (3.32).

4.2. Evolution of the spectral transform

The free entry $\Omega^+ = \Omega^+(k,t)$ is the *dispersion relation* and, according to the general theory [20], it is strictly dependent on the choice of solution. It is called a dispersion relation because, in the infinite line case, it coincides with the dispersion law of the linearized evolution. The differential equation (4.7) is evaluated at $x = 0$ and $x \to \infty$. This provides 8 equations for the 4 components of $\Omega^+(k,t)$ and the 4 evolution equations of $\rho(k,t)$, $\tau(k,t)$ and their conjugates (these evolutions are compatible, which actually results from the reduction). We obtain

$$\Omega^+(k,t) = -\frac{\mathrm{i}}{4}\mathcal{I}(k)\left[2\rho(k,t)\overline{A}B\sigma_3 + \begin{pmatrix} |A|^2 - |B|^2 & 0 \\ 2\overline{A}B & -|A|^2 + |B|^2 \end{pmatrix}\right] \tag{4.8}$$

(note that the dispersion relation depends on the spectral transform itself), and the Riccati evolution

$$\rho_t = \frac{\mathrm{i}}{2}\mathcal{I}\left[\,\overline{A}B\,\rho^2 - (|A|^2 - |B|^2)\,\rho - A\overline{B}\,\right] \tag{4.9}$$

where we have defined

$$\mathcal{I}(k) = \int \frac{g(\lambda)\,\mathrm{d}\lambda}{\lambda - k} \qquad \mathrm{Im}\,(k) > 0. \tag{4.10}$$

The main fact is that the above evolution conserves the Laurent expansion of $\rho(k,t)$. Indeed, seeking a solution of (4.9) as the power series

$$\rho(k,t) = \sum_{j=1}^{p} \rho_j(t)k^j + \sum_{n=0}^{\infty} \frac{1}{k^n}\rho^{(n)}(t) \tag{4.11}$$

allows us to prove by induction that $\rho_j(t) = 0$ for $j \geqslant 2$. For $j = 1$ we have the following evolution equation:

$$\frac{\rho_{1t}}{\rho_1^2} = \frac{\mathrm{i}}{2} A\overline{B} \int \mathrm{d}\lambda\; g(\lambda) = \alpha(t) \tag{4.12}$$

which is solved to get

$$\rho_1(t) = \frac{\rho_1(0)}{1 - \rho_1(0)\int_0^t \alpha(t)}. \tag{4.13}$$

Then, if ρ_1 vanishes at time $t = 0$, it vanishes for all t. Such is also the case for $\rho^{(0)}$ whose evolution (with $\rho_1(t) = 0$) is simply

$$\rho_t^{(0)} = 0. \tag{4.14}$$

Consequently, as ρ_1 and $\rho^{(0)}$ vanish at initial time (because $\rho(k,0)$ is the spectral transform of the Dirichlet datum $q(x,0)$), they vanish for all time and the solution of the Riccati equation (4.9) possesses the right behaviour for large k, compatible with (2.26).

The coefficient of k^{-1} gives rise to

$$\partial_t \rho^{(1)}(t) = \frac{\mathrm{i}}{2} A\overline{B} \int \mathrm{d}\lambda\; g(\lambda). \tag{4.15}$$

The evolution (4.1) written in $x = 0$ gives

$$\partial_t q_0(t) = -A\overline{B} \int \mathrm{d}\lambda\; g(\lambda) \tag{4.16}$$

and moreover we have, by construction,

$$\rho^{(1)}(0) = \frac{q_0(0)}{2\mathrm{i}}. \tag{4.17}$$

Consequently the unique solution to (4.15) is

$$\rho^{(1)}(t) = \frac{q_0(t)}{2\mathrm{i}} \tag{4.18}$$

as required by (2.26).

The time evolution (4.9) of the spectral transform has coefficients depending only on the input boundary values $A(t)$ and $B(t)$ and containing all the relevant information on the nonlinear scattering of the three waves q, a and b. In some cases this Riccati equation is explicitly solvable, for instance when $A(t)$ and $B(t)$ are proportional (which is physically meaningful) as illustrated in [16]. It can then be checked in the explicit expression of $\rho(k,t)$ of [16] that it does obey the large k asymptotics. The fact of being able to provide an explicit expression of the spectral transform in a class of physically interesting cases makes the present approach to boundary value problems very interesting.

4.3. Why it works

The Dirichlet open-end problem is well posed if the solution $q(x,t)$ constructed above is proved to obey, for all t, the boundary value

$$q(x,t)|_{x=0^+} = q_0(t) \tag{4.19}$$

and to belong to the class of functions such that

$$\lim_{x\to\infty} q(x,t) = 0. \tag{4.20}$$

The proof proceeds along the following steps.

(1) The solution $\rho(k,t)$ of the Riccati equation (4.9) possesses the large k expansion

$$\rho(k,t) = \frac{q_0(t)}{2\mathrm{i}k} + \mathcal{O}(1/k^2) \tag{4.21}$$

resulting from (4.18). This result is central to the theory as we see here that the time evolution of the spectral transform conserves the properties it has at $t = 0$, see (2.26). Note that the expansion can be pursued at any order.

(2) With such a behaviour of $\rho(k,t)$ the property P.3 still holds, namely

$$\Psi(k,0,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.22}$$

The proof of the above boundary value is obtained by following exactly the method of section 3 where q_0 has simply to be understood as $q_0(t)$.

(3) Compute the limit of $q(x,t)$ as $x \to 0^+$ from expression (3.27) by following the method of section 3 used to prove property P.8.

(4) Compute the limit of $q(x,t)$ as $x \to \infty$ from expression (3.27), again by following the proof of property P.8. Note that the essential property to be used is the fact that $\rho(k,t)$ is continuous and bounded for $k \in \mathbb{R}$ at any given t. Particular attention is required when, as t evolves, a pole of ρ is *'seen'* to cross the real axis. It has been demonstrated in [18] that the process is continuous, or put in other words, a resonance (pole in the lower half-plane) disappears and a pole appears while never belonging to the real axis. The same situation arises for the Schrödinger eigenvalue problem [25].

4.4. Summary of the method of solution of SRS

The solution of SRS with Dirichlet condition is achieved by the following steps.

(1) The spectral transform at time zero $\rho(k,0)$ is obtained from the Dirichlet condition $\tilde{q}(x)$ by finding the vector solution $\phi_2(k,x,0)$ of the integral equations (2.4) in the upper half k-plane, namely

$$\begin{aligned} \phi_{12}^+(k,x,0) &= -\int_x^\infty \mathrm{d}y\, \tilde{q}(y)\phi_{22}^+(k,y,0)\mathrm{e}^{-2\mathrm{i}k(x-y)} \\ \phi_{22}^+(k,x,0) &= 1 - \int_0^x \mathrm{d}y\, \overline{\tilde{q}(y)}\phi_{12}^+(k,y,0) \end{aligned} \tag{4.23}$$

and then $\rho(k,0)$ results from

$$\rho(k,0) = \phi_{12}^+(k,x,0)\big|_{x=0^+}. \tag{4.24}$$

(2) $\rho(k,t)$ is calculated in the upper half k-plane from $\rho(k,0)$ by solving the Riccati time evolution

$$\rho_t = \frac{\mathrm{i}}{2}\mathcal{I}\big[\, \overline{A}B\,\rho^2 - (|A|^2 - |B|^2)\,\rho - A\overline{B}\,\big] \tag{4.25}$$

with entries $A(t)$ and $B(t)$. The N poles k_n in the upper half-plane and related residues ρ_n of $\rho(k, t)$ must be identified for each value of the parameter t.

(3) The integral system (3.5) and (3.6) which, within the reduction (3.34), simplifies to

$$\psi_{11}^{+}(k, x, t) = 1 + \frac{1}{2\mathrm{i}\pi} \int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda - k}\, \overline{\rho(\bar{\lambda}, t)} \mathrm{e}^{2\mathrm{i}\lambda x} \psi_{12}^{-}(\lambda, x, t) + \sum_{n=1}^{N(t)} \frac{\overline{\rho_n(t)}}{\overline{k_n(t)} - k} \psi_{12}^{-}(\overline{k_n(t)}, x, t) \mathrm{e}^{2\mathrm{i}\overline{k_n(t)}x} \tag{4.26}$$

$$\psi_{12}^{-}(k, x, t) = \frac{1}{2\mathrm{i}\pi} \int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda - k}\, \rho(\lambda, t) \mathrm{e}^{-2\mathrm{i}\lambda x} \psi_{11}^{+}(\lambda, x, t) - \sum_{n=1}^{N(t)} \frac{\rho_n(t)}{k_n(t) - k} \psi_{11}^{+}(k_n, x, t) \mathrm{e}^{-2\mathrm{i}k_n(t)x} \tag{4.27}$$

is solved to get $\psi_{11}^{+}(k, x, t)$. This system of equations holds for any fixed value of the *'parameters'* x and t.

(4) Finally the solution $q(x, t)$ is obtained from expression (3.27), i.e.

$$q(x, t) = -\frac{1}{\pi} \int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \mathrm{d}\lambda\, \rho(\lambda, t) \psi_{11}^{+}(\lambda, x, t) \mathrm{e}^{-2\mathrm{i}\lambda x} + 2\mathrm{i} \sum_{n=1}^{N(t)} \rho_n(t) \psi_{11}^{+}(k_n(t), x, t) \mathrm{e}^{-2\mathrm{i}k_n(t)x}. \tag{4.28}$$

5. The Dirichlet problem for sine-Gordon

5.1. Boundary values and Lax pair

Another case where the method can be applied successfully is the SG equation in the light cone:

$$\theta_{xt} + \sin\theta = 0 \tag{5.1}$$

which is solved here for Dirichlet conditions $\theta_0(t)$ and $\tilde{\theta}(x)$, namely

$$t \in [0, T] : \quad \theta(0, t) = \theta_0(t) \qquad x \in [0, \infty) : \quad \theta(x, 0) = \tilde{\theta}(x) \tag{5.2}$$

in the class of functions vanishing (modulo 2π) with all derivatives as $x \to \infty$ for every value of time t (the value T is arbitrary).

This equation is fundamental in many areas of physics and has been consequently widely studied. The solution of the boundary value problem on the quadrant has also been the subject of intense studies, see e.g. [8–10, 13, 14] and the references therein.

The Lax pair, found in [29], is given by the Zakharov–Shabat equation for a particular choice of potentials r and q, namely

$$\Phi_x = \mathrm{i}k[\Phi, \sigma_3] + Q\Phi \qquad Q = -\frac{\mathrm{i}}{2}\sigma_2\theta_x \tag{5.3}$$

together with the following evolution:

$$\Phi_t = -\frac{\mathrm{i}}{4k}(\sigma_3\cos\theta + \sigma_1\sin\theta)\,\Phi + \Phi \mathrm{e}^{-\mathrm{i}k\sigma_3 x}\Omega \mathrm{e}^{\mathrm{i}k\sigma_3 x}. \tag{5.4}$$

We have chosen to write this Lax pair for the particular solution Φ defined in section 2. Note that the above particular structure of the potential Q obeys the reduction constraint (3.36).

5.2. Evolution of the spectral transform

By using the behaviours (2.8) and (2.9) in (5.4) we calculate Ω and get the time evolution of the spectral transform. Within the reduction relations (3.38) we eventually obtain

$$\Omega = -\frac{\mathrm{i}}{4k}\begin{pmatrix} \cos\theta_0 - \rho(-k,t)\sin\theta_0 & 0 \\ 0 & -\cos\theta_0 + \rho(k,t)\sin\theta_0 \end{pmatrix} \tag{5.5}$$

$$\rho_t = \frac{\mathrm{i}}{4k}\left[\rho^2 \sin\theta_0 - 2\rho\cos\theta_0 - \sin\theta_0\right]. \tag{5.6}$$

Note that the reduction constraint $\overline{\rho}(-\overline{k}) = \rho(k)$ ensures the real value potential $\theta(x,t)$ is conserved by the above time evolution (as soon as $\theta_0 \in \mathbb{R}$ of course).

The evolution of $\rho(k,t)$ is of Riccati type and depends exclusively on the boundary value $\theta_0(t)$, while the other Dirichlet condition $\tilde{\theta}(x)$ fixes the initial value $\rho(k,0)$. Following exactly the same method as for SRS we show that, as $\rho_1(0)$ and $\rho^{(0)}(0)$ vanish (by construction), the large k asymptotic behaviour of the solution $\rho(k,t)$ is

$$\rho(k,t) = \frac{1}{k}\rho^{(1)} + \mathcal{O}(1/k^2) \qquad \partial_t\rho^{(1)} = -\frac{\mathrm{i}}{4}\sin\theta_0. \tag{5.7}$$

Thanks to the SG equation evaluated in $x = 0$ we can integrate the above time evolution and obtain

$$\rho^{(1)}(t) = -\frac{1}{4\mathrm{i}}\theta_x(0,t) = \frac{q_0(t)}{2\mathrm{i}} \tag{5.8}$$

as indeed, by construction of $\rho(k,0)$ from $\tilde{\theta}(x)$, we have

$$\rho^{(1)}(0) = -\frac{1}{4\mathrm{i}}\left.\tilde{\theta}_x\right|_{x=0} = -\frac{1}{4\mathrm{i}}\theta_x(0,0). \tag{5.9}$$

The last point to consider is the apparent singularity of $\rho(k,t)$ when $k \to 0$ by real values. Actually, for $k \to 0$ the spectral transform possesses the following regular behaviour:

$$\rho(k,t) = r_0(t) + kr_1(t) + \mathcal{O}(k^2) \tag{5.10}$$

that can be deduced from the very definition (2.10) of $\rho(k,t)$ (or equivalently by (5.18) below), by expanding these expressions in powers of k. Inserting the above expansion in (5.6) gives the following unique solution vanishing for $\theta_0 = 0$:

$$\rho(0,t) = r_0(t) = \frac{\cos\theta_0 - 1}{\sin\theta_0}. \tag{5.11}$$

The other terms $r_n(t)$ in the expansion are then recursively given in terms of r_j, $(j < n)$, without constraint.

5.3. Why it works

The Dirichlet open-end problem is well posed if the solution $\theta(x,t)$ constructed above is proved to obey, for all t, the boundary value

$$\theta(x,t)|_{x=0^+} = \theta_0(t) \tag{5.12}$$

and to belong to the class of functions such that

$$\lim_{x\to\infty}\theta_x(x,t) = 0. \tag{5.13}$$

The proof proceeds along the following steps.

(1) The solution $\rho(k,t)$ of the Riccati equation (5.6) possesses the large k expansion

$$\rho(k,t) = \frac{q_0(t)}{2\mathrm{i}k} + \mathcal{O}(1/k^2) \tag{5.14}$$

where we have

$$q_0(t) = -\tfrac{1}{2}\tilde{\theta}_x(0) + \tfrac{1}{2}\int_0^t \sin\theta_0 \tag{5.15}$$

resulting from (5.7).

(2) Then property P.3 holds for all t, namely

$$\Psi(k, 0, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.16}$$

The proof is obtained by following the method of section 3 where q_0 has simply to be understood as $q_0(t)$, given from Dirichlet conditions by (5.15).

(3) The limit of $q(x, t)$ as $x \to 0^+$ is computed from expression (3.27) by following the method of section 3 used to prove property P.8.

(4) The limit of $q = -\theta_x/2$ as $x \to \infty$ is computed from expression (3.27), again by following the proof of property P.8. Note that the essential property to be used is the fact that $\rho(k, t)$ is continuous and bounded for $k \in \mathbb{R}$ at any given t.

5.4. Summary of the method of solution of sine-Gordon

The solution of SG with Dirichlet conditions is achieved by the following steps.

(1) The spectral transform at time zero, $\rho(k, 0)$, is obtained from the Dirichlet condition $\tilde{\theta}(x)$ by finding the vector solution $\phi_2(k, x, 0)$ of the integral equations (2.4) in the upper half k-plane, namely by solving

$$\begin{aligned} \phi_{12}^+(k, x, 0) &= -\int_x^\infty \mathrm{d}y\, \tilde{q}(y)\phi_{22}^+(k, y, 0)\mathrm{e}^{-2\mathrm{i}k(x-y)} \\ \phi_{22}^+(k, x, 0) &= 1 + \int_0^x \mathrm{d}y\, \tilde{q}(y)\phi_{12}^+(k, y, 0) \end{aligned} \tag{5.17}$$

where $\tilde{q} = -\tilde{\theta}_x/2$. Then $\rho(k, 0)$ results from

$$\rho(k, 0) = \phi_{12}^+(k, x, 0)\big|_{x=0^+}. \tag{5.18}$$

(2) The spectral transform at time t, $\rho(k, t)$, is calculated in the upper half k-plane from $\rho(k, 0)$ and Dirichlet condition $\theta_0(t)$ by solving for $k \neq 0$ the Riccati time evolution:

$$\rho_t = \frac{\mathrm{i}}{4k}\left[\rho^2 \sin\theta_0 - 2\rho\cos\theta_0 - \sin\theta_0\right] \tag{5.19}$$

and for $k = 0$

$$\rho(0, t) = \frac{\cos\theta_0 - 1}{\sin\theta_0}. \tag{5.20}$$

The N poles k_n in the upper half-plane and related residues ρ_n of $\rho(k, t)$ must be identified for each value of the parameter t.

(3) The integral system (3.5) and (3.6) which, within the reduction (3.38), simplifies to

$$\begin{aligned} \psi_{11}^+(k, x, t) = 1 &- \frac{1}{2\mathrm{i}\pi}\int_{-\infty-\mathrm{i}0}^{+\infty-\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda - k}\, \rho(-\lambda, t)\mathrm{e}^{2\mathrm{i}\lambda x}\psi_{12}^-(\lambda, x, t) \\ &- \sum_{n=1}^{N(t)} \frac{\rho_n(t)}{k_n(t) + k}\psi_{12}^-(-k_n(t), x, t)\mathrm{e}^{-2\mathrm{i}k_n(t)x} \end{aligned} \tag{5.21}$$

$$\begin{aligned} \psi_{12}^-(k, x, t) = &\frac{1}{2\mathrm{i}\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \frac{\mathrm{d}\lambda}{\lambda - k}\, \rho(\lambda, t)\mathrm{e}^{-2\mathrm{i}\lambda x}\psi_{11}^+(\lambda, x, t) \\ &- \sum_{n=1}^{N(t)} \frac{\rho_n(t)}{k_n(t) - k}\psi_{11}^+(k_n, x, t)\mathrm{e}^{-2\mathrm{i}k_n(t)x} \end{aligned} \tag{5.22}$$

is solved to get $\psi_{11}^{+}(k, x, t)$. This system of equations holds for any fixed value of the *'parameters'* x and t.

(4) Finally the solution $\theta(x,t)$ is obtained in the form $\theta_x(x,t) = -2q(x,t)$, where $q(x,t)$ results from ψ_{11}^{+} and ρ as

$$q(x,t) = -\frac{1}{\pi}\int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \mathrm{d}\lambda\, \rho(\lambda,t)\psi_{11}^{+}(\lambda,x,t)\mathrm{e}^{-2\mathrm{i}\lambda x} + 2\mathrm{i}\sum_{n=1}^{N(t)} \rho_n(t)\psi_{11}^{+}(k_n(t),x,t)\mathrm{e}^{-2\mathrm{i}k_n(t)x}. \tag{5.23}$$

5.5. An explicit example

To illustrate the above method we consider the explicitly solvable case of a piecewise constant boundary datum $\theta_0(t)$, namely

$$\begin{aligned} t \in [0, t_1] &: \quad \theta_0(t) = 0 \\ t \in [t_j, t_{j+1}] &: \quad \theta_0(t) = \varphi_j \\ t \in [t_n, \infty] &: \quad \theta_0(t) = 0 \end{aligned} \tag{5.24}$$

compatible with a vanishing initial condition

$$\theta(x,0) = \tilde{\theta}(x) = 0. \tag{5.25}$$

Note that such a boundary datum represents an arbitrary localized function $\theta_0(t)$ discretized with an n-step time grid.

We adopt the following notation:

$$\begin{aligned} &\rho_j(t) = \rho(k,t) \qquad \text{for} \quad t \in [t_j, t_{j+1}] \\ &\rho_j = \rho_j(t_{j+1}) \equiv \rho(k, t_{j+1}). \end{aligned} \tag{5.26}$$

Then the solution of the evolution (5.6) can be written for $t \in [t_j, t_{j+1}]$:

$$\rho_j(t) = -\frac{b_j(\rho_{j-1}+a_j)\mathrm{e}^{\mathrm{i}(t-t_j)/2k} - a_j(\rho_{j-1}+b_j)}{(\rho_{j-1}+a_j)\mathrm{e}^{\mathrm{i}(t-t_j)/2k} - (\rho_{j-1}+b_j)} \tag{5.27}$$

with the following definitions:

$$a_j = -\frac{1+\cos\varphi_j}{\sin\varphi_j} \qquad b_j = \frac{1-\cos\varphi_j}{\sin\varphi_j}. \tag{5.28}$$

Note that the limit $\varphi_j \to 0$ from (5.27) does produces the solution to evolution (5.6) where $\theta_0 = \varphi_j$ is set to zero.

The large-k behaviour of solution (5.27) is obtained by induction:

$$\rho_j(t) = -\frac{\mathrm{i}}{4k}\left[(t-t_j)\sin\varphi_j + \sum_{\ell=1}^{j}(t_\ell - t_{\ell-1})\sin\varphi_{\ell-1}\right] + \mathcal{O}\left(\frac{1}{k^2}\right). \tag{5.29}$$

Hence we have

$$\partial_t\rho_j(t) = -\frac{\mathrm{i}}{4k}\sin\varphi_j + \mathcal{O}\left(\frac{1}{k^2}\right) \tag{5.30}$$

compatible with (5.7) and solution (5.27) has the expected large-k behaviour.

In the last step, that is for $\varphi_n = 0$, (5.27) can be written

$$\rho_n(t) = -\frac{(b_n\rho_{n-1}-1)\mathrm{e}^{\mathrm{i}(t-t_n)/2k} - a_n\rho_{n-1}+1}{(\rho_{n-1}+a_n)\mathrm{e}^{\mathrm{i}(t-t_n)/2k} - (\rho_{n-1}+b_n)} \tag{5.31}$$

which eventually simplifies to

$$\rho_n(t) = \rho_{n-1}\mathrm{e}^{\mathrm{i}(t-t_n)/2k}. \tag{5.32}$$

As a consequence, the spectral transform evaluated *after* the total effect of the boundary datum $\theta_0(t)$, i.e. for $t > t_n$, is completely determined by $\rho_{n-1} = \rho(k, t_n)$. This function of k is itself obtained as the following *mapping*:

$$\rho_0 = 0$$
$$\rho_j = -\frac{b_j(\rho_{j-1}+a_j)\mathrm{e}^{\mathrm{i}(t_{j+1}-t_j)/2k} - a_j(\rho_{j-1}+b_j)}{(\rho_{j-1}+a_j)\mathrm{e}^{\mathrm{i}(t_{j+1}-t_j)/2k} - (\rho_{j-1}+b_j)} \qquad j = 1, \ldots, n-1. \tag{5.33}$$

This is an explicit expression that allows us to get an insight into the effect of a generic input datum $\theta_0(t)$ on a vanishing background under suitable (arbitrary) discretization. We report the study of the above mapping (in the complex k-plane) in forthcoming work and simply mention here that the main point concerns the study of the creation (annihilation) of poles of $\rho(k, t)$ in the upper half k-plane.

Remembering the reality constraint (3.40), poles of $\rho(k)$ come in pairs, and a pole $k = k_n(t)$ of $\rho_j(t)$ is given by

$$\mathrm{e}^{\mathrm{i}\frac{t-t_j}{2k_n}} = \frac{\rho_{j-1}+b_j}{\rho_{j-1}+a_j} \tag{5.34}$$

where ρ_{j-1} is itself a function of k_n.

This equation for the pole k_n is in general highly complicated but, if $\rho_{j-1} = 0$ (as for the initial stage $t = t_1$), the above relation becomes

$$\mathrm{e}^{\mathrm{i}\frac{t-t_1}{2k_n}} = -\frac{1-\cos\varphi_1}{1+\cos\varphi_1}. \tag{5.35}$$

Defining

$$\gamma = \ln\frac{1-\cos\varphi_1}{1+\cos\varphi_1} \tag{5.36}$$

the solution of (5.35) can be written

$$k_n(t) = \frac{1}{2}\left[(2n+1)\pi + \mathrm{i}\gamma\right]\frac{(t-t_1)}{(2n+1)^2\pi^2+\gamma^2} \tag{5.37}$$

where n assumes values in $\mathbb{Z}$. The above expression is identical to the one obtained in [13] (see formula 5.52) for $t - t_1 = T$. A property of our approach is to explictly display the *time dynamics* of the spectral transform (such as pole generation and motion).

Among the above set of poles, only those with positive imaginary parts contribute to the soliton spectrum. It is then remarkable that

$$\varphi_1 \in \left[0, \frac{\pi}{2}\right[\;\Rightarrow\; \gamma < 0 \;\Rightarrow\; \mathrm{Im}\,(k_n) < 0 \tag{5.38}$$

$$\varphi_1 \in \left]\frac{\pi}{2}, \pi\right] \;\Rightarrow\; \gamma > 0 \;\Rightarrow\; \mathrm{Im}\,(k_n) > 0 \tag{5.39}$$

which shows that no poles are generated by a consant boundary datum $\theta_0(t) = \varphi_1$ smaller than $\pi/2$, while an infinity of them are generated for $\varphi_1 > \pi/2$. An example of pole locations is shown in figure 1 where we see that the point $k = 0$ is an accumulation point for the infinity of poles. Indeed when $n \to \infty$ we have from (5.37) $k_n \to 0$. The case $\varphi_1 = \pi/2$ is singular as $\rho(k, t)$ would then possess poles on the real axis, for which the theory fails.

Note that the discretization of a smooth localized boundary datum $\theta_0(t)$ produces *small* φ_j's. Hence the soliton generation would then result from the contributions in equation (5.34) of $\rho_{j-\ell}$. In the case of a single constant boundary datum of amplitude greater than $\pi/2$, the assigning of solitons to the presence of an infinity of poles is not clear (usually an infinity of poles is related to a self-similar structure).

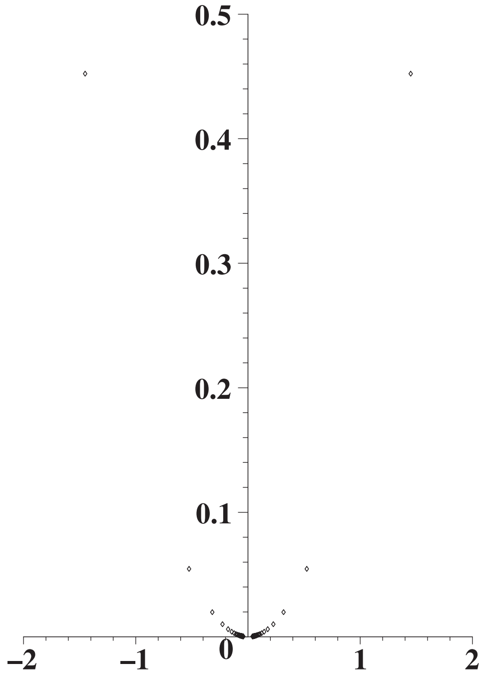

Figure 1. Positions of the poles of $\rho(k,t)$ for $n \in [-20, 19]$, $t - t_1 = 10$ and $\varphi_1 = 0.65\pi$.

Acknowledgments

It is a pleasure to aknowledge enlightening discussions with M Boiti, E V Doktorov, A V Mikhailov, B Pelloni, F Pempinelli, P C Sabatier and V S Shchesnovich. This work has been partially suported by contract INTAS 99-1782.

References

[1] Gardner C S, Green J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095
[2] Zakharov V E and Shabat A B 1972 *Sov. Phys.–JETP* **37** 62
[3] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
[4] Sklyanin E K 1987 *Funct. Anal. Appl.* **21** 164
[5] Bibikov P N and Tarasov V O 1989 *Teor. Math. Phys.* **79** 570
[6] Bikbaev R F and Tarasov V O 1991 *J. Phys. A: Math. Gen.* **24** 2507
[7] Sakhnovich L A 1991 *Ukr. Math. J.* **43** 1470
Sakhnovich L A 1992 *Russ. Math. Izv. Vyssh. Uchebn. Zaved.* **36** 42
[8] Bikbaev R F and Tarasov V O 1991 *Algebr. Anal.* **3** 775
[9] Belokolos E D 1995 *Theor. Math. Phys.* **103** 613
[10] Fokas A S and Its A R 1992 *Phys. Rev. Lett.* **68** 3117
Fokas A S and Its A R 1996 *SIAM J. Math. Anal.* **27** 738
[11] Sabatier P C 2000 *J. Math. Phys.* **41** 414
[12] Boutet de Monvel A and Kotlyarov V 2000 *Inverse Problems* **16** 1813
[13] Fokas A S 1997 *Proc. R. Soc.* A **453** 1411
Fokas A S 2000 *J. Math. Phys.* **41** 4188
[14] Pelloni B 2002 *Commun. Appl. Anal.* **6** 179
[15] Fokas A S and Pelloni B 1998 *Proc. R. Soc.* A **454** 645
Fokas A S and Pelloni B 2000 *Phys. Rev. Lett.* **84** 4785
[16] Leon J and Mikhailov A V 1999 *Phys. Lett.* A **253** 33
[17] Fokas A S and Menyuk C R 1999 *J. Nonlinear Sci.* **9** 313

[18] Boiti M, Caputo J-G, Leon J and Pempinelli F 2000 *Inverse Problems* **16** 303
[19] Boiti M, Leon J and Pempinelli F 1999 *J. Phys. A: Math. Gen.* **32** 927–43
[20] Leon J 1994 *J. Math. Phys.* **35** 3054
Leon J 1993 *Phys. Rev.* A **47** 3264
Leon J 1992 *Phys. Lett.* A **170** 283
[21] Claude C, Ginovart F and Leon J 1995 *Phys. Rev.* A **52** 767
[22] Doktorov E V and Shchesnovich V S 2001 Nearly integrable nonlinear equations on the half-line *Inverse Problems* **17** 971
(Doktorov E V and Shchesnovich V S 2000 *Preprint* arXiv:nlin.SI/0010002)
[23] Claude C and Leon J 1995 *Phys. Rev. Lett.* **74** 3479
[24] Hugot F-X and Leon J 1999 *Inverse Problems* **15** 701
[25] Hugot F-X and Leon J 2001 *J. Phys. A: Math. Gen.* **34** 117
[26] Levitan B M and Sargsjan I S 1975 *Introduction to Spectral Theory (Transl. Math. Monographs, Am. Math. Soc.)* (Providence, RI: American Mathematical Society)
Iliev I D, Khristov E Kh and Kirchev K P 1994 *Spectral Methods in Soliton Equations (Pitman Monographs)* (Harlow: Longman)
[27] Mushkhelishvili N I 1958–72 *Singular Integral Equations* (Groningen Nederland: Wolters-Noordhoff)
[28] Chu F Y F and Scott A C 1975 *Phys. Rev.* A **12** 2060
[29] Ablowitz M J, Kaup D J, Newell A C and Segur H 1973 *Phys. Rev. Lett.* **30** 1262
Takhtadzhyan L A and Faddeev L D 1974 *Theor. Math. Phys.* **21** 1046